

Definability of types over finite partial order indiscernibles

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Abstract

In this paper, we show that a partitioned formula φ is dependent if and only if φ has uniform definability of types over finite partial order indiscernibles. This generalizes our result from a previous paper [1]. We show this by giving a decomposition of the truth values of an externally definable formula on a finite partial order indiscernible.

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1. Introduction

In [1], we introduce the notion of uniform definability of types over finite sets (UDTFS) and conjecture that all dependent formulas have UDTFS (we call this the UDTFS Conjecture). In that paper, we approach a solution to the conjecture from two distinct directions. First, we take a subclass of the class of dependent theories and show that this subclass has UDTFS; namely, we show that all dp-minimal theories have UDTFS. We hope to show this for larger subclasses in future papers. Our second approach involves slightly weakening the definition of UDTFS. In the first section of [1], we actually give a characterization of dependent formulas in terms of definability of types. Theorem 1.2 (ii) of [1] states that a formula is dependent if and only if it has uniform definability of types over finite indiscernible sequences.

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Indiscernible sequences are very strong and well behaved in the context of dependent theories, so this result is not too surprising. On the other hand, as one continues to weaken the assumption of “indiscernible sequence,” one gets closer to solving the UDTFS Conjecture. In this paper, we generalize Theorem 1.2 (ii) of [1] using generalized indiscernible sequences. We prove the following theorem:

Theorem 1.1. *The following are equivalent for a partitioned formula $\varphi(\bar{x}; \bar{y})$:*

- (i) *φ is dependent;*
- (ii) *there exists a formula $\psi(\bar{y}; \bar{z}_0, \dots, \bar{z}_{n-1})$ such that, for all finite partial orders $(P; \leq)$, all (generalized) indiscernibles $\langle \bar{b}_i : i \in P \rangle$ (see Definition 3.1 below), and all types $p \in S_\varphi(\{\bar{b}_i : i \in P\})$, there exists $i_0, \dots, i_{n-1} \in P$ so that, for all $i \in P$, $\varphi(\bar{x}; \bar{b}_i) \in p(\bar{x})$ if and only if $\models \psi(\bar{b}_i; \bar{b}_{i_0}, \dots, \bar{b}_{i_{n-1}})$.*

That is, we show that a formula φ is dependent if and only if it has uniform definability of types over finite partial order indiscernibles. The notion of generalized indiscernibles is first introduced in Chapter VII of [2]. As in the work of Scow [3], this paper characterizes dependence in terms of generalized indiscernible sequences. However, in this paper, we use partial order indiscernibles instead of ordered graph indiscernibles.

If we can push this to its natural conclusion, we could solve the UDTFS Conjecture. For example, suppose that φ has independence dimension $\leq n$ and we took as our index language $S = \{P_\eta : \eta \in {}^{n+1}2\}$ for $(n+1)$ -ary predicates P_η . Then φ has UDTFS if and only if it has uniform definability of types over finite S -structure indiscernibles. Thus, we view Theorem 1.1 as a definite step toward solving the UDTFS Conjecture.

For this paper, a “formula” will mean a \emptyset -definable formula in a fixed language L unless otherwise specified. If $\theta(\bar{x})$ is a formula, then let me denote $\theta(\bar{x})^0 = \neg\theta(\bar{x})$ and $\theta(\bar{x})^1 = \theta(\bar{x})$. We will be working in a complete, first-order theory T in a fixed language L with monster model \mathfrak{C} . Fix $M \models T$ (so $M \preceq \mathfrak{C}$) and a partitioned L -formula $\varphi(\bar{x}; \bar{y})$. By $\varphi(M; \bar{b})$ for some $\bar{b} \in \mathfrak{C}^{\text{lg}(\bar{y})}$, we mean the following subset of $M^{\text{lg}(\bar{x})}$:

$$\varphi(M; \bar{b}) = \{\bar{a} \in M^{\text{lg}(\bar{x})} : \models \varphi(\bar{a}; \bar{b})\}.$$

We will say that a set $B \subseteq \mathfrak{C}^{\text{lg}(\bar{y})}$ is φ -independent if, for any map $s : B \rightarrow 2$, the set of formulas $\{\varphi(\bar{x}; \bar{b})^{s(\bar{b})} : \bar{b} \in B\}$ is consistent. We will say that φ

has *independence dimension* $N < \omega$, which we will denote by $\text{ID}(\varphi) = N$, if N is maximal such that there exists $B \subseteq \mathfrak{C}^{\text{lg}(\bar{y})}$ with $|B| = N$ where B is φ -independent. We will say that φ is dependent (some authors call this NIP for “not the independence property”) if $\text{ID}(\varphi) = N$ for some $N < \omega$. Finally, we will say that a theory T is dependent if all partitioned formulas are dependent.

Fix a set of partitioned formulas $\Phi(\bar{x}; \bar{y}) = \{\varphi_i(\bar{x}; \bar{y}) : i \in I\}$. By a “ Φ -type over B ” for some set B of $\text{lg}(\bar{y})$ -tuples we mean a consistent set of formulas of the form $\varphi_i(\bar{x}; \bar{b})^t$ for some $t < 2$ and ranging over all $\bar{b} \in B$ and $i \in I$. If p is a Φ -type over B , then we will say that p has domain $\text{dom}(p) = B$. For any B a set of $\text{lg}(\bar{y})$ -tuples, the space of all Φ -types with domain B is denoted $S_\Phi(B)$. If $\Phi = \{\varphi\}$ is a singleton, then we will replace φ with $\{\varphi\}$ in our previous definitions.

The remainder of this paper is organized as follows: In Section 2, we focus only on partial orders. Abstractifying the notion of indiscernibility to simply colorings on partial orders, we produce a means of partitioning the ordering into homogeneous subsets with respect to the coloring. Applying this to indiscernibles, this generalizes the “bounded alternation rank” characterization of dependent formulas (Theorem II.4.13 (2) of [2]) and may be of independent interest. In Section 3, we define and discuss partial order indiscernibles. We prove Theorem 1.1 above using the techniques of [1] and the partitioning theorem from Section 2. Finally, in Section 4, we discuss the broader implications of this result and state some natural open questions that remain.

2. Partial Orders

Before discussing general indiscernibility and even model theory, we first work in the universe of pure partial orders. The discussion before Dilworth’s Theorem (Theorem 2.3) is elementary and the results are certainly not due to this author.

Fix $(P; \leq)$ a partial order. An *antichain* A of the partial order P is a subset of P such that, for all $i, j \in A$, $i \not\leq j$ and $j \not\leq i$. By contrast, a *chain* C of the partial order P is a subset of P such that, for all $i, j \in C$, $i \leq j$ or $j \leq i$. We will say that an antichain A is *maximal* if there does not exist $i \in P - A$ such that $A \cup \{i\}$ is an antichain, and we similarly define a *maximal* chain. For any subset $P_0 \subseteq P$, $(P_0; \leq|_{P_0 \times P_0})$ is a partial order and will be called a *suborder* of $(P; \leq)$. For any maximal antichain A , define the following sets:

(i) $D(A) = \{i \in P : (\exists j \in A)(i \triangleleft j)\}$ (the downward closure of A).

(ii) $U(A) = \{i \in P : (\exists j \in A)(j \triangleleft i)\}$ (the upward closure of A).

Lemma 2.1. *Fix $A \subseteq P$ a maximal antichain. For any $i \in P - A$, exactly one of the following hold:*

(i) *There exists $j \in A$ such that $i \triangleleft j$.*

(ii) *There exists $j \in A$ such that $j \triangleleft i$.*

That is, $\{D(A), A, U(A)\}$ is a partition of P .

Proof. By the maximality of A and transitivity. □

In general, for any antichain A , if $j \triangleleft i$ for some $i \in A$, then $i \not\triangleleft j$ for all $i \in A$. In this case, we say that $j \triangleleft A$. If there exists $i \in A$ such that $i \triangleleft j$, we say that $A \triangleleft j$. If $j \not\triangleleft A$ and $A \not\triangleleft j$, then $A \cup \{j\}$ is again an antichain. If $j \triangleleft A$ or $j \in A$, we write $j \trianglelefteq A$ and similarly for $A \trianglelefteq j$.

Lemma 2.2. *Fix $A \subseteq P$ a maximal antichain and suppose $A' \subseteq (D(A) \cup A)$ is an antichain. Then there exists A'' with $A' \subseteq A'' \subseteq (D(A) \cup A)$ such that A'' is a maximal antichain of P (the whole order).*

Proof. If A' is not a maximal antichain of P , then there exists $j \in P - A'$ such that $A' \cup \{j\}$ is an antichain. If $j \notin (D(A) \cup A)$, then $j \in U(A)$ by Lemma 2.1. Therefore, there exists $i \in A$ so that $i \triangleleft j$. Since $A' \subseteq (D(A) \cup A)$, either $i \in A'$, $A' \triangleleft i$, or $A' \cup \{i\}$ is an antichain. If $i \in A'$ or $A' \triangleleft i$, then $A' \triangleleft j$, contrary to assumption. Therefore, $A' \cup \{i\}$ is an antichain contained in $(D(A) \cup A)$ that is \triangleleft -below j . Use Zorn's Lemma to conclude. □

This also holds for $U(A) \cup A$ by symmetry. Given two maximal antichains $A, A' \subseteq P$, say that $A \trianglelefteq A'$ if $A \subseteq D(A') \cup A'$ (i.e., for all $i \in A$, $i \in A'$ or $i \triangleleft A'$). Of course, there can be maximal antichains that are incomparable, but transitivity will clearly hold for this relation. Furthermore, for any maximal antichain $A \subseteq P$, if $D(A) \neq \emptyset$, then there exists $A' \subseteq P$ a maximal antichain such that $A' \triangleleft A$ (and similarly if $U(A) \neq \emptyset$) by Lemma 2.2.

For any $A \trianglelefteq A'$ maximal antichains of P , define $(A, A') = U(A) \cap D(A')$, let $[A, A') = (A \cup U(A)) \cap D(A')$, and let $[A, A'] = A \cup (A, A') \cup A'$. Let $[-\infty, A) = D(A)$, let $[-\infty, A] = A \cup D(A)$, let $[A, \infty) = A \cup U(A)$, and let $[-\infty, \infty) = P$ (think of these as “intervals” of P). So, for any $A_0 \triangleleft A_1 \triangleleft \dots \triangleleft A_n$

maximal antichains of P , $[-\infty, A_0)$, $[A_0, A_1)$, ..., $[A_{n-1}, A_n)$, $[A_n, \infty)$ is a partition of P .

We define $\text{Lev}_n^-(P)$, the n th level of P from below, by induction as follows:

$$\text{Lev}_n^-(P) = \left\{ i \in P - \bigcup_{\ell < n} \text{Lev}_\ell^-(P) : \left(\nexists j \in P - \bigcup_{\ell < n} \text{Lev}_\ell^-(P) \right) (j \triangleleft i) \right\}.$$

So $\text{Lev}_0^-(P)$ is the antichain of the least elements of P , and $\text{Lev}_1^-(P)$ is the antichain of the least elements of $P - \text{Lev}_0^-(P)$, and so on. We define $\text{Lev}_n^+(P)$ by reversing the ordering. Notice that, for all $i \in \text{Lev}_n^-(P)$, there exists $i_0 \in \text{Lev}_0^-(P)$, ..., $i_{n-1} \in \text{Lev}_{n-1}^-(P)$ such that $i_0 \triangleleft \dots \triangleleft i_{n-1} \triangleleft i$.

Theorem 2.3 (Dilworth's Theorem, [4]). *Fix $n < \omega$. If $(P; \trianglelefteq)$ is a finite partial order such that, for all antichains $A \subseteq P$, $|A| \leq n$, then P is the disjoint union of at most n chains.*

We now discuss 2-colorings of a finite partial order $(P; \trianglelefteq)$. We will use this in the next section when proving definability of types over finite partial order indiscernibles.

Definition 2.4. Fix $(P; \trianglelefteq)$ a partial order, $f : P \rightarrow \{0, 1\}$, and $N < \omega$. We say that f is a N -indiscernible coloring of P if,

- (i) for all antichains $A \subseteq P$, there exists $t < 2$ such that $|\{i \in P : f(i) = t\}| \leq N$; and
- (ii) there does not exist $i_0 \triangleleft i_1 \triangleleft \dots \triangleleft i_{2N+1}$ from P such that, for all $\ell < 2N + 1$, $f(i_\ell) = 1 - f(i_{\ell+1})$ (that is, the coloring on any chain does not alternate more than $2N + 1$ times).

Fix $(P; \trianglelefteq)$ a finite partial order and $f : P \rightarrow 2$ a N -indiscernible coloring of P . For any subset $X \subseteq P$ and $t < 2$, define X^t as follows:

$$X^t = \{i \in X : f(i) = t\} = (f^{-1}(t) \cap X).$$

Note that $X = X^0 \cup X^1$ and $X^0 \cap X^1 = \emptyset$. For any antichain $A \subseteq P$ with $|A| > 2N$, there exists a unique $t < 2$ such that $|A^t| \leq N$. If not, then A would violate condition (i) of Definition 2.4. In this case, define $\text{Maj}(A) = t$ (Maj stands for “majority”). We now use this to give a means of breaking down partial orders P in terms of subsets X on which f is constant.

Lemma 2.5. *Let $M = (2N + 1)(N + 1)$. There exists $A_0 \triangleleft \dots \triangleleft A_{K-1}$ for $K \leq 2N + 2$ maximal antichains of P such that, for all $n \leq K$ and all antichains $A \subseteq [A_{n-1}, A_n)$, $|A^{n(\bmod 2)}| \leq M$ (let $A_{-1} = -\infty$ and $A_K = \infty$). That is, each $P_n = [A_{n-1}, A_n)$ is such that all antichains $A \subseteq P_n$ have $f(i) = n + 1(\bmod 2)$ for “almost all” $i \in A$.*

Proof. We inductively construct, for each n , $A_n \subseteq P$ a maximal antichain of P as follows: Fix $n \geq 0$ and suppose A_ℓ are defined for all $\ell < n$. If it exists, choose $A_n \subseteq P$ maximal such that

- (i) $A_0 \triangleleft \dots \triangleleft A_{n-1} \triangleleft A_n$,
- (ii) $|A_n^{n(\bmod 2)}| > M$ (hence $\text{Maj}(A_n) \equiv n(\bmod 2)$), and
- (iii) A_n is \triangleleft -minimal such.

If no such A_n exists, set $K = n$ and the construction terminates. For any $n \leq K$ and any antichain $A \subseteq [A_{n-1}, A_n)$, if $|A^{n(\bmod 2)}| > M$, then extend this to a maximal antichain $A' \subseteq [A_{n-1}, A_n]$ (which exists by Lemma 2.2 on the suborder $P_0 = (-\infty, A_n]$). Then $A_{n-1} \triangleleft A' \triangleleft A_n$ and $|(A')^{n(\bmod 2)}| > M$, contrary to the minimality of A_n . Therefore, $|A^{n(\bmod 2)}| \leq M$. We now show that this process terminates in $K < 2N + 2$ steps.

Assuming $K \geq 2N + 2$, inductively define, for each $n < K$, $A_n^* \subseteq A_n^{n(\bmod 2)}$ with

- (i) $|A_n^*| > (2N + 1 - n)(N + 1)$, and
- (ii) for all $i \in A_n^*$, there exists $i_0 \in A_0^*, \dots, i_{n-1} \in A_{n-1}^*$ such that $i_0 \triangleleft i_1 \triangleleft \dots \triangleleft i_{n-1} \triangleleft i$.

Let $A_0^* = (A_0)^0$, which satisfies (i) by assumption and (ii) vacuously. Now, suppose that A_{n-1}^* is constructed. For each $X \subseteq A_{n-1}^*$ with $|X| = N + 1$ and $Y \subseteq A_n^{n(\bmod 2)}$ with $|Y| = N + 1$, we claim that there exists $i \in X$ and $j \in Y$ such that $i \triangleleft j$. If not, then $i \not\triangleleft j$ for all such i, j . However, since $X \subseteq A_{n-1}^* \subseteq P^{n+1(\bmod 2)}$ and $Y \subseteq A_n^{n(\bmod 2)}$, we see that $i \neq j$. Furthermore, since $A_{n-1} \triangleleft A_n$, $j \not\triangleleft i$. Therefore $X \cup Y$ is an antichain. However, this contradicts Definition 2.4 (i). Therefore, choosing $i_0 \in X$ and $j_0 \in Y$ such that $i_0 \triangleleft j_0$, we consider now $(N + 1)$ -element subsets of $A_{n-1}^* - \{i_0\}$ and $(N + 1)$ -element subsets of $A_n^{n(\bmod 2)} - \{j_0\}$. Continuing in this manner, we see that there exists $A_n^* \subseteq A_n^{n(\bmod 2)}$ such that each $i \in A_n^*$ is \triangleleft -below some

element of A_{n-1}^* and $|A_n^*| > (2N+1-n)(N+1)$. Thus A_n^* satisfies conditions (i) and (ii), as desired.

Finally, consider A_{2N+1}^* . By (i), $|A_{2N+1}^*| > 0$, so it is, in particular, non-empty. Fix $i_{2N+1} \in A_{2N+1}^*$. By condition (ii), there exists $i_0 \in A_0^*, \dots, i_{2N} \in A_{2N}^*$ such that $i_0 \triangleleft i_1 \triangleleft \dots \triangleleft i_{2N} \triangleleft i_{2N+1}$. However, for each $\ell < 2N+2$, since $i_\ell \in A_\ell^* \subseteq P^{\ell \pmod{2}}$, $f(i_\ell) \equiv \ell \pmod{2}$. This contradicts Definition 2.4 (ii). Therefore, $K < 2N+2$, as desired. \square

Theorem 2.6 (*N-Indiscernible Coloring Decomposition Theorem*). *Let $M = (2N+1)(N+1)$. There exists $A_0 \triangleleft \dots \triangleleft A_{K-1}$ for $K \leq 2N+2$ maximal antichains of P and $C_{n,\ell} \subseteq [A_{n-1}, A_n)$ for $\ell < M$ chains of P such that*

$$P^1 = \left(\bigcup_{n \equiv 0 \pmod{2}} [A_{n-1}, A_n) - \left(\bigcup_{\ell < M} C_{n,\ell} \right) \cup \bigcup_{n \equiv 1 \pmod{2}, \ell < M} C_{n,\ell} \right).$$

Proof. Use the maximal antichains $A_0, \dots, A_{K-1} \subseteq P$ as given by Lemma 2.5. Fix $n \leq K$ and consider $P_n = \{i \in [A_{n-1}, A_n) : f(i) \equiv n \pmod{2}\}$. By the condition given in Lemma 2.5, for each antichain A of (P_n, \trianglelefteq) , $|A| \leq M$. Therefore, by Theorem 2.3, (P_n, \trianglelefteq) is the disjoint union of at most M chains, say $C_{n,\ell}$. That is, $P_n = \bigcup_{\ell < M} C_{n,\ell}$. Therefore, for each n and each $i \in [A_{n-1}, A_n)$, $f(i) \equiv n \pmod{2}$ if and only if $i \in \bigcup_{\ell < M} C_{n,\ell}$. The conclusion follows. \square

There are two problems with this decomposition in terms of uniform definability. For one, the antichains A_n may be arbitrarily large, so checking if $i \in [A_{n-1}, A_n)$ could require arbitrarily much information from A_{n-1} and A_n . Another problem is that the chains $C_{n,\ell}$ may be arbitrarily large. We address the problems in reverse order. First, for any $i_0, i_1 \in P$, define $[i_0, i_1]_P = \{i \in P : i_0 \trianglelefteq i \trianglelefteq i_1\}$.

Lemma 2.7. *Fix $t < 2$ and suppose that $C \subseteq P^t$ is any chain. There exists $i_0 \trianglelefteq i'_0 \trianglelefteq i_1 \trianglelefteq i'_1 \trianglelefteq \dots \trianglelefteq i_K \trianglelefteq i'_K$ from C for $K \leq N$ such that*

$$C \subseteq \left(\bigcup_{n \leq K} [i_n, i'_n]_P \right) \subseteq P^t.$$

Proof. Let i_0 be the minimal element of C and i'_K the maximal element of C . For every $j \in [i_0, i_K]_P$ such that $f(j) = 1 - t$, let $i_j^- \triangleleft i_j^+$ be from C so

that $i_j^- \triangleleft j \triangleleft i_j^+$ and i_j^- is \triangleleft -maximal such and i_j^+ is \triangleleft -minimal such. Fix $n \geq 0$ and suppose that i_n is constructed. Choose $j \in ([i_0, i_K]_P)^{1-t}$ such that $i_n \trianglelefteq i_j^-$, i_j^- is \triangleleft -minimal such, and i_j^+ is \triangleleft -minimal such. Let $i'_n \in C$ be \triangleleft -maximal such that $i'_n \triangleleft i_j^+$ and let $i_{n+1} = i_j^+$. If no such j exists, then the construction terminates and set $K = n$.

First, it is clear that $C \subseteq \bigcup_{n < K} [i_n, i'_n]_P$ as, for each $n < K$, i'_n and i_{n+1} are C -consecutive. We claim that, $\bigcup_{n < K} [i_n, i'_n]_P \subseteq P^t$ and $K \leq N$. First, fix $j \in [i_n, i'_n]_P$. If $j = i_n$ or $j = i'_n$, then $f(j) = t$, so we may assume $j \neq i_n, i'_n$. If $f(j) = 1-t$, then we have $i_n \triangleleft j \triangleleft i'_n$. Therefore $i_n \trianglelefteq i_j^-$ and $i_j^+ \trianglelefteq i'_n$, contrary to construction. Therefore, $f(j) = t$. Secondly, suppose that $K > N$. By construction, for any $n < N$, there exists $j_n \in P^{1-t}$ such that $i_n \triangleleft j_n \triangleleft i_{n+1}$. Hence, we see that $i_0 \triangleleft j_0 \triangleleft i_1 \triangleleft j_1 \triangleleft \dots \triangleleft i_N \triangleleft j_N$ contradicts Definition 2.4 (ii). This gives the desired result. \square

So the chains $C_{n,\ell}$ can be taken to be a union of at most $N + 1$ closed intervals. What about the arbitrarily large antichains?

Lemma 2.8. *For all maximal antichains A and both $t < 2$ such that $|A^{1-t}| \leq N$, there exists $A_0 \subseteq A$ with $|A_0| \leq 2N + 1$, $J^- \subseteq P^{1-t}$ with $|J^-| \leq N$, and $J^+ \subseteq P^{1-t}$ with $|J^+| \leq N$ so that, for all $j \in P^{1-t}$,*

(i) $j \triangleleft A$ if and only if $j \triangleleft A_0$ or $j \trianglelefteq J^-$, and

(ii) $A \triangleleft j$ if and only if $A_0 \triangleleft j$ or $J^+ \trianglelefteq j$.

Proof. If $|A| \leq 2N$, then set $A_0 = A$, $J^- = J^+ = \emptyset$. So we may assume $|A| > 2N$ and $t = \text{Maj}(A)$. Fix any subset $A_0 \subseteq A$ such that $|A_0^t| = N + 1$ and $A_0^{1-t} = A^{1-t}$. Since $t = \text{Maj}(A)$, $|A^{1-t}| \leq N$. Therefore, $|A_0| \leq 2N + 1$. We now construct J^- by induction as follows: Let $J_0^- = \emptyset$. Fix $n > 0$ and suppose that J_{n-1}^- is constructed. If there exists $j \in P^{1-t}$ such that $j \triangleleft A$ and $A_0 \cup J_{n-1}^- \cup \{j\}$ is an antichain, then choose $j' \triangleleft A$ \triangleleft -maximal such that $j \trianglelefteq j'$. Then, $A_0 \cup J_{n-1}^- \cup \{j'\}$ is clearly still an antichain (if j' were below some $i \in A_0 \cup J_{n-1}^- \cup \{j'\}$, then j would be too by transitivity). Let $J_n^- = J_{n-1}^- \cup \{j'\}$ and continue. The construction halts when there exist no such j and we set $J^- = J_{n-1}^-$. Construct J^+ similarly for $j \in P^{1-t}$ so that $A \triangleleft j$.

We claim that this construction works and (each) halts in at most N steps. Since $A_0 \cup J_n^-$ is an antichain, $|A_0^t| = N + 1$, and $J^- \subseteq P^{1-t}$, if $|J^-| > N$, then this contradicts Definition 2.4 (i). Therefore, $|J^-| \leq N$. If

$j \in P^{1-t}$ and $j \triangleleft A$, then either $j \triangleleft A_0$, or $j \trianglelefteq J^-$ by construction of J^- and similarly for J^+ . \square

This lemma implies that maximal antichains form a strong barrier for the non-majority color. That is, the relationship of any $j \in D(A)^{1-t}$ to all of $[A, \infty)$ is determined by a set of size $\leq 3N + 1$.

Corollary 2.9. *Fix A a maximal antichain and $t < 2$ such that $|A^{1-t}| \leq N$. For A_0 and J^- as in Lemma 2.8, there exists a partition of $[A, \infty)$ into X_I for $I \subseteq (A_0 \cup J^-)$ such that, for all $j \in P^{1-t}$ with $j \triangleleft A$, for all $i \in [A, \infty)$, $j \triangleleft i$ if and only if $i \in X_{\{i' \in A_0 : j \triangleleft i'\} \cup \{j' \in J^- : j \trianglelefteq j'\}}$.*

Proof. Fix A'_0 and J^- given as in Lemma 2.8. For $i \in [A, \infty)$, put $i \in X_I$ if and only if $I = \{i' \in A_0 \cup J^- : i' \trianglelefteq i\}$. \square

A similar result holds for $A_0 \cup J^+$ and $[-\infty, A]$ by symmetry. We will use Theorem 2.6 and the other tools of this section in the next section to prove Theorem 1.1.

3. General Δ -Indiscernibility

3.1. Introduction

Work in a complete theory T in a language L with monster model \mathfrak{C} . Fix $\Delta(\bar{z}_0, \dots, \bar{z}_n)$ any set of L -formulas where $\text{lg}(\bar{z}_i) = \text{lg}(\bar{z}_j)$ and let P be an S -structure for some different language, S (we will call this language S the *index language*). Let $\langle \bar{b}_i : i \in P \rangle$ be a sequence of elements from $\mathfrak{C}^{\text{lg}(\bar{z}_0)}$ indexed by P .

Definition 3.1 (General Indiscernibility). The sequence $\langle \bar{b}_i : i \in P \rangle$ is Δ -*indiscernible* (with respect to the S -structure P) if, for all $i_0, \dots, i_n \in P$ distinct and all $j_0, \dots, j_n \in P$ distinct such that $\text{qftp}_S(i_0, \dots, i_n) = \text{qftp}_S(j_0, \dots, j_n)$ (i.e. there exists a partial S -elementary map f so that $f(i_k) = j_k$ for all $k \leq n$),

$$\text{tp}_\Delta(\bar{b}_{i_0}, \dots, \bar{b}_{i_n}) = \text{tp}_\Delta(\bar{b}_{j_0}, \dots, \bar{b}_{j_n}).$$

If we drop the Δ , then we mean that $\langle \bar{b}_i : i \in P \rangle$ is Δ -indiscernible for all appropriate Δ .

In this section, we will be interested in the case where Δ is finite, $S = \{\trianglelefteq\}$, and P is a partial order. In the case where P is a linear order, Definition 3.1 is the usual definition of a Δ -indiscernible sequence. When P is completely unordered, Definition 3.1 is the usual definition of a Δ -indiscernible set. Given $(P; \trianglelefteq)$ a partial order, a sequence $\langle \bar{b}_i : i \in P \rangle$ is Δ -indiscernible if and only if, for all $i_0, \dots, i_n \in P$ distinct and all $j_0, \dots, j_n \in P$ distinct, if $i_k \trianglelefteq i_\ell$ if and only if $j_k \trianglelefteq j_\ell$ for all $k, \ell \leq n$, then

$$\text{tp}_\Delta(\bar{b}_{i_0}, \dots, \bar{b}_{i_n}) = \text{tp}_\Delta(\bar{b}_{j_0}, \dots, \bar{b}_{j_n}).$$

Suppose now that $\varphi(\bar{x}; \bar{y})$ is any dependent formula. For any $n < \omega$, define

$$\Delta_{n, \varphi}(\bar{z}_0, \dots, \bar{z}_n) = \left\{ \exists \bar{x} \left(\bigwedge_{i \leq n} \varphi(\bar{x}; \bar{z}_i)^{s(i)} \right) : s \in {}^{n+1}2 \right\}. \quad (1)$$

Using the machinery of indiscernible sequences with this special set of formulas $\Delta_{n, \varphi}$, we aim to prove the following theorem:

Theorem 3.2. *The following are equivalent for a partitioned formula $\varphi(\bar{x}; \bar{y})$:*

(i) φ is dependent.

(ii) *There exists $N, K, L < \omega$ and formulas $\psi_\ell(\bar{y}; \bar{z}_0, \dots, \bar{z}_K)$ for $\ell < L$ such that, for all finite partial orders $(P; \trianglelefteq)$, all $\Delta_{N, \varphi}$ -indiscernible sequences $\langle \bar{b}_i : i \in P \rangle$, and all $p \in S_\varphi(\{\bar{b}_i : i \in P\})$, there exists $\ell < L$ and $i_0, \dots, i_K \in P$ such that, for all $j \in P$,*

$$\varphi(\bar{x}; \bar{b}_j) \in p(\bar{x}) \text{ if and only if } \models \psi_\ell(\bar{b}_j; \bar{b}_{i_0}, \dots, \bar{b}_{i_K}).$$

As an immediate corollary, we get Theorem 1.1. That is, a partitioned formula $\varphi(\bar{x}; \bar{y})$ is dependent if and only if it has uniform definability of types over finite partial order indiscernibles. This generalizes the result of Theorem 1.2 (ii) of [1].

First, to show that (ii) implies (i), we only need to count types. Suppose, by means of contradiction, that φ is independent and (ii) holds. Then, by Ramsey's Theorem, for any $m < \omega$, there exists $\langle \bar{b}_i : i \in P \rangle$ a $\Delta_{N, \varphi}$ -indiscernible sequence, where $(P; \trianglelefteq)$ is a linear order with $|P| = m$, such that the set $\{\bar{b}_i : i \in P\}$ is φ -independent. Therefore, the size of $S_\varphi(\{\bar{b}_i : i \in P\})$ is exactly 2^m . However, since each type in $S_\varphi(\{\bar{b}_i : i \in P\})$ is determined by $\ell < L$ and $i_0, \dots, i_K \in P$, the number of φ -types over $\{\bar{b}_i : i \in P\}$ is

$\leq L \cdot |P|^K = L \cdot m^K$. Therefore, $2^m \leq L \cdot m^K$. However, our choice of m was arbitrary (in particular, independent of L and K). This is a contradiction.

The converse is trickier to show, and will involve a detailed analysis of $\Delta_{N,\varphi}$ -indiscernible sequences. Suppose φ is dependent and let $N = \text{ID}(\varphi)$. Let $\Delta = \Delta_{N,\varphi}$ as in (1) above. We begin with a lemma for Δ -indiscernible sequences indexed by partial orders. The proof of this lemma is a simple modification of the proof of Theorem II.4.13 of [2], but we include it here for completeness.

Lemma 3.3. *Fix $(P; \trianglelefteq)$ a partial order, let $\langle \bar{b}_i : i \in P \rangle$ be a Δ -indiscernible sequence, and fix any $\bar{a} \in \mathfrak{C}^{\text{lg}(\bar{x})}$. Let $f : P \rightarrow 2$ be defined by, for all $i \in P$, $f(i) = 1$ if and only if $\models \varphi(\bar{a}; \bar{b}_i)$. Then f is an N -indiscernible coloring of P .*

Proof. (i): Suppose, by way of contradiction, that there exists $i_0^0, \dots, i_N^0, i_0^1, \dots, i_N^1 \in A$ distinct from some antichain $A \subseteq P$ such that $f(i_\ell^t) = t$ for all $t < 2$ and $k \leq N$. Then, for any $s \in {}^{N+1}2$, the following formula is witnessed by \bar{a} :

$$\models \exists \bar{x} \left(\bigwedge_{\ell \leq N} \varphi(\bar{x}; \bar{b}_{i_\ell^{s(\ell)}})^{s(\ell)} \right).$$

However, by Δ -indiscernibility, we get that

$$\models \exists \bar{x} \left(\bigwedge_{\ell \leq N} \varphi(\bar{x}; \bar{b}_{i_\ell^0})^{s(\ell)} \right).$$

Since this holds for all $s \in {}^{N+1}2$, we get that $\{\bar{b}_{i_\ell^0} : \ell \leq N\}$ is a φ -independent set of size $N + 1$, contrary to the fact that $\text{ID}(\varphi) = N$.

(ii): Suppose, by way of contradiction, that we have $i_0 \triangleleft \dots \triangleleft i_{2N+1}$ such that $f(i_\ell) \neq f(i_{\ell+1})$ for all $\ell < 2N + 1$. Without loss of generality, suppose $f(\bar{b}_{i_0}) = 0$. For any $s \in {}^{N+1}2$, as witnessed by \bar{a} , we have that

$$\models \exists \bar{x} \left(\bigwedge_{\ell \leq N} \varphi(\bar{x}; \bar{b}_{i_{2\ell+s(\ell)}})^{s(\ell)} \right).$$

By Δ -indiscernibility, we get that

$$\models \exists \bar{x} \left(\bigwedge_{\ell \leq N} \varphi(\bar{x}; \bar{b}_{i_{2\ell}})^{s(\ell)} \right).$$

Again, this yields a contradiction. \square

By Theorem 2.6 and the tools of Section 2, to prove the remainder of Theorem 3.2, it suffices to show that the ordering of P is L -definable. For the remainder of this section, fix $(P; \trianglelefteq)$ a finite partial order, $\langle \bar{b}_i : i \in P \rangle$ a Δ -indiscernible sequence with respect to P , and $\bar{a} \in \mathfrak{C}^{\text{lg}(\bar{x})}$. As in the previous section, for any $X \subseteq P$ and $t < 2$, define

$$X^t = \{i \in X : \models \varphi(\bar{a}; \bar{b}_i)^t\}.$$

Our method for defining types will be to use Theorem 2.6 to decompose P , then use uniform definitions for handling the various pieces. For large subsets $X \subseteq P$, we cannot hope to get exact definitions for which \bar{b}_i are such that $i \in X$ with a bounded number of parameters. Instead, we will focus on “rough definitions.” Fix $t < 2$ and $X \subseteq P^t$. We say that X is *roughly definable* if there exists $\gamma_X(\bar{y})$ uniform over boundedly many elements of $\{\bar{b}_i : i \in P\}$ such that

- (i) For all $i \in X$, $\models \gamma_X(\bar{b}_i)$, and
- (ii) For all $i \in P$, if $\models \gamma_X(\bar{b}_i)$, then $i \in P^t$.

If we can break up, for some $t < 2$, P^t into a bounded number of subsets X_0, \dots, X_n , each of which is roughly definable, then we can uniformly define which $i \in P^t$, hence develop a uniform definition of finite φ -types. This will be the goal of the remainder of this section.

3.2. Homogeneous Sets

One useful tool will be homogeneity.

Definition 3.4. We say that $X \subseteq P$ is *homogeneous* (with respect to the Δ -indiscernible sequence $\langle \bar{b}_i : i \in P \rangle$) if, for all $i_0, \dots, i_N \in X$ distinct and all $j_0, \dots, j_N \in X$ distinct,

$$\text{tp}_\Delta(\bar{b}_{i_0}, \dots, \bar{b}_{i_N}) = \text{tp}_\Delta(\bar{b}_{j_0}, \dots, \bar{b}_{j_N}).$$

That is, $\langle \bar{b}_i : i \in X \rangle$ is Δ -indiscernible over the empty structure on X .

For example, any antichain $A \subseteq P$ is homogeneous. In the next few lemmas, we will show how to define large homogeneous subsets of P . The following lemma is shown exactly as Lemma 3.3 (i), noting that the only fact we used about A was that it was homogeneous:

Lemma 3.5. *For any $X \subseteq P$ homogeneous, there exists $t < 2$ such that $|X^t| \leq N$.*

So the point now will be to start with some homogeneous set X with a majority color $t < 2$. Then, add on elements of P^{1-t} , preserving homogeneity, until this is no longer possible. By Lemma 3.5, we can add no more than N elements from P^{1-t} while still preserving homogeneity. However, we will need to insure that the homogeneity of a large set X is determined by a bounded subset $X_0 \subseteq X$. We accomplish this by using Lemma 2.8.

Fix $A \sqsubseteq A'$ two maximal antichains and $t < 2$ so that $|A^t| > N$ and $|(A')^t| > N$. Let A_0 and J^- be given by Lemma 2.8 for A and let A'_0 and J^+ be given by Lemma 2.8 for A' . Let $J_0 = A_0 \cup J^-$ and $J_1 = A'_0 \cup J^+$ (so $|J_0| \leq 2N + 1$ and $|J_1| \leq 2N + 1$). Therefore, by Lemma 2.8, for all $j \in P^{1-t} - [A, A']$, $j \sqsubseteq J_0$ or $J_1 \sqsubseteq j$. We now partition $[A, A']$ according to how it relates to J_0 and J_1 (similarly to Corollary 2.9). For all $J \subseteq J_0$ and $J' \subseteq J_1$, define

$$X_{J,J'} = \{i \in [A, A'] : (\forall j \in J_0)(j \sqsubseteq i \leftrightarrow j \in J) \wedge (\forall j' \in J_1)(i \sqsubseteq j' \leftrightarrow j' \in J')\}.$$

Lemma 3.6. *Fix any two antichains $A_0, A_1 \in [A, A']$ and suppose that, for all $J \subseteq J_0$ and $J' \subseteq J_1$, either*

- (i) $|A_0 \cap X_{J,J'}| = |A_1 \cap X_{J,J'}|$, or
- (ii) $|A_0 \cap X_{J,J'}| > N$ and $|A_1 \cap X_{J,J'}| > N$.

Then, for any $I_0 \subseteq P^{1-t} - [A, A']$, $A_0 \cup I_0$ is homogeneous if and only if $A_1 \cup I_0$ is homogeneous.

Proof. Fix A_0 and A_1 as above and $I_0 \subseteq P^{1-t} - [A, A']$. Since the conditions are symmetric, suppose that $A_0 \cup I_0$ is homogeneous and we show that $A_1 \cup I_0$ is homogeneous. Fix any $i_0, \dots, i_N \in A_1 \cup I_0$ distinct and we will define $i'_\ell \in A_0 \cup I_0$ inductively on $\ell \leq N$ so that the map $i_\ell \mapsto i'_\ell$ is an isomorphism of S -substructures. First, if $i_\ell \in I_0$, then set $i'_\ell = i_\ell$. Otherwise, fix J, J' so that $i_\ell \in X_{J,J'}$ and choose $i'_\ell \in A_0 \cap X_{J,J'} - \{i'_0, \dots, i'_{\ell-1}\}$. This exists by assumptions (i) or (ii). Since A_0 and A_1 are both antichains, there is no relationship amongst the elements there. For any $j \in I_0$, $j \triangleleft i_\ell$ (for $i_\ell \in X_{J,J'}$) if and only if $(\forall j' \in J_0)(j \sqsubseteq j' \leftrightarrow j' \in J)$ for $j \sqsubseteq J_0$ and likewise for $J_1 \sqsubseteq j$ and J' . But this holds if and only if $j \triangleleft i'_\ell$ as i_ℓ and i'_ℓ belong to the same $X_{J,J'}$. Therefore, $i_\ell \mapsto i'_\ell$ is an isomorphism and, by Δ -indiscernibility,

$$\text{tp}_\Delta(\bar{b}_{i_0}, \dots, \bar{b}_{i_N}) = \text{tp}_\Delta(\bar{b}_{i'_0}, \dots, \bar{b}_{i'_N}).$$

Since $A_0 \cup I_0$ is homogeneous, this implies that $A_1 \cup I_0$ is homogeneous. \square

Corollary 3.7. *Fix A a maximal antichain and $t < 2$ such that $|A^{1-t}| \leq N$. There exists $A_0 \subseteq A$, with $|A_0| \leq (N+1)2^{(2N+2)}$, such that, for any $I_0 \subseteq P^{1-t}$, $A \cup I_0$ is homogeneous if and only if $A_0 \cup I_0$ is homogeneous.*

Proof. If $|A| \leq 2N$, set $A_0 = A$ and we are done. So we may assume $|A| > 2N$ and $|A^t| > N$. Use Lemma 3.6 on $A = [A, A]$. Then indeed any subset of $A = [A, A]$ is an antichain. For each J, J' , choose $A_{J,J'}$ maximal in $X_{J,J'}$ such that $|A_{J,J'}| \leq N$. Then, taking $A_0 = \bigcup_{J,J'} A_{J,J'}$ suffices. \square

Lemma 3.8. *If $t < 2$ and $A \subseteq P^t$ is an antichain, then there exists a uniform formula $\gamma_A(\bar{y})$, over at most $N + (N+1)2^{(2N+2)}$ elements of $\{\bar{b}_i : i \in P\}$, such that*

- (i) *for all $i \in A$, $\models \gamma_A(\bar{b}_i)$, and*
- (ii) *for all $i \in P$, if $\models \gamma_A(\bar{b}_i)$, then $i \in P^t$.*

That is, antichains are roughly definable.

Proof. If $|A| \leq N$, set $\gamma_A(\bar{y}) = \bigvee_{i \in A} \bar{y} = \bar{b}_i$. So we may assume $|A| > N$, hence we can expand it to a maximal antichain $A' \subseteq P$ where $\text{Maj}(A') = t$. Fix A_0 as in Corollary 3.7 and choose $I_0 \subseteq P^{1-t}$ so that $A_0 \cup I_0$ is homogeneous and I_0 is maximal such. By Lemma 3.5, $|I_0| \leq N$.

Now, for any $j \in (P^{1-t} - (A_0 \cup I_0))$, since $A_0 \cup I_0 \cup \{j\}$ is not homogeneous, there exists $i_0, \dots, i_{N-1}, i_N \in A_0 \cup I_0$ distinct and $i'_0, \dots, i'_{N-1} \in A_0 \cup I_0$ distinct such that

$$\text{tp}_\Delta(\bar{b}_{i_0}, \dots, \bar{b}_{i_N}) \neq \text{tp}_\Delta(\bar{b}_{i'_0}, \dots, \bar{b}_{i'_{N-1}}, \bar{b}_j).$$

Therefore, there exists $\delta \in \pm\Delta$ so that

$$\models \delta(\bar{b}_{i_0}, \dots, \bar{b}_{i_N}) \wedge \neg\delta(\bar{b}_{i'_0}, \dots, \bar{b}_{i'_{N-1}}, \bar{b}_j).$$

However, fix any $i \in A' - A_0$. Then, by Corollary 3.7,

$$\text{tp}_\Delta(\bar{b}_{i_0}, \dots, \bar{b}_{i_N}) = \text{tp}_\Delta(\bar{b}_{i'_0}, \dots, \bar{b}_{i'_{N-1}}, \bar{b}_i),$$

hence $\models \delta(\bar{b}_{i'_0}, \dots, \bar{b}_{i'_{N-1}}, \bar{b}_i)$. Thus, the formula $\delta(\bar{b}_{i'_0}, \dots, \bar{b}_{i'_{N-1}}, \bar{y})$ distinguishes j from all $i \in A - A_0$. Let

$$\begin{aligned} \gamma'(\bar{y}) = \bigwedge \Big\{ & \delta(\bar{b}_{i_0}, \dots, \bar{b}_{i_{N-1}}, \bar{y}) : \delta \in \pm\Delta, i_0, \dots, i_{N-1} \in A_0 \cup I_0, \\ & \models \delta(\bar{b}_{i_0}, \dots, \bar{b}_{i_{N-1}}, \bar{b}_{i^*}) \text{ for some } i^* \in (A_0 \cup I_0) - \{i_0, \dots, i_{N-1}\} \Big\}. \end{aligned}$$

Finally, let

$$\gamma_A(\overline{y}) = \left(\gamma'(\overline{y}) \wedge \bigwedge_{i \in A_0^{1-t} \cup I_0} \overline{y} \neq \overline{b}_i \right) \vee \bigvee_{i \in A_0^t} \overline{y} = \overline{b}_i.$$

Then, for all $i \in A$, either $i \in A_0^t$ and clearly $\models \gamma_A(\overline{b}_i)$, or $i \in A - A_0 \subseteq A' - A_0$, in which case $\models \gamma'(\overline{b}_i)$ by construction. Therefore, condition (i) holds. Similarly, if $j \in P^{1-t}$, then either $j \in A_0^{1-t} \cup I_0$ and clearly $\models \neg \gamma_A(\overline{b}_j)$, or $j \notin A_0^{1-t} \cup I_0$, in which case $\models \neg \gamma'(\overline{b}_j)$ by construction. This gives us condition (ii). \square

In the next two subsections, we break the problem up into two cases depending on whether or not chains are homogeneous. As in the previous section, let $M = (2N + 1)(N + 1)$.

3.3. When Chains are Homogeneous

For this subsection, we assume:

Case 1. For any $i_0 \triangleleft \dots \triangleleft i_N$ from P , for all $\sigma \in S_{N+1}$ (the group of permutations on $N + 1$), we have that

$$\text{tp}_\Delta(\overline{b}_{i_0}, \dots, \overline{b}_{i_N}) = \text{tp}_\Delta(\overline{b}_{i_{\sigma(0)}}, \dots, \overline{b}_{i_{\sigma(N)}}).$$

That is, chains are homogeneous. Thus, for any chain $C \subseteq P$, there exists $t < 2$ such that $|C^t| \leq N$.

Lemma 3.9. *Under the assumption of Case 1, there exists $t < 2$ such that P^t is a union of $\leq M(2N + 2)$ chains and $\leq N(2N + 2)$ antichains.*

Proof. Let $A_0, \dots, A_{K-1} \subseteq P$ as given by Theorem 2.6 and let $P_n = [A_{n-1}, A_n)$ (where $A_{-1} = -\infty$ and $A_K = \infty$). We already have that, for all $n \leq K$, $P_n^{n(\bmod 2)}$ is a union of $\leq M$ chains by Theorem 2.6. Fix $n \leq K$ minimal such that $P_n^{(n+1)(\bmod 2)}$ is not equal to a union of $\leq N$ antichains and $\leq M$ chains. If $n = K$, then we can take $t = K(\bmod 2)$ (as each P_n^t is a union of $\leq M$ chains or $\leq M$ chains and $\leq N$ antichains). So we may assume that $n < K$. We claim that $t = n(\bmod 2)$ still works.

Consider the antichains $\text{Lev}_\ell^-(P_n^{1-t})$ for $\ell < N$ and let $P_n^* = P_n^{1-t} - (\bigcup_{\ell < N} \text{Lev}_\ell^-(P_n^{1-t}))$. Since P_n^{1-t} is not the union of $\leq N$ antichains (for example, $\text{Lev}_\ell^-(P_n^{1-t})$ for $\ell < N$) and $\leq M$ chains, P_n^* cannot be the union

of $\leq M$ chains. By Theorem 2.3, there exists $A \subseteq P_n^*$ an antichain with $|A| > M$.

Now choose $A^* \subseteq P$ a maximal antichain with $|(A^*)^t| > M$, $A_n \triangleleft A^*$, and A^* \triangleleft -maximal such. This exists since $n < K$. Thus, $(A^*, \infty)^t$ is a union of $\leq M$ antichains and, for all $n' \leq n$, $P_{n'}^t$ is a union of $\leq M$ chains and $\leq N$ antichains. Thus, it suffices to check this condition for $[A_n, A^*]^t$.

Let $P^{**} = [A_n, A^*]^t - \bigcup_{\ell < N} \text{Lev}_\ell^+([A_n, A^*]^t)$. We claim that P^{**} has no antichain of size $> M$. If it did, then fix $A' \subseteq P^{**}$ such an antichain. By construction, $|A| > M$, $|A'| > M$, $A^{1-t} = A$, and $(A')^t = A'$. Therefore, by Lemma 3.3 (i), there exists $i \in A$ and $i' \in A'$ such that $i \triangleleft i'$ or $i' \triangleleft i$. However, $i \triangleleft A_n$ and $A_n \trianglelefteq i'$. Therefore, $i \triangleleft i'$. By the definition of levels, there exists $i_\ell \in \text{Lev}_\ell^-(P_n^{1-t})$ such that $i_0 \triangleleft i_1 \triangleleft \dots \triangleleft i_{N-1} \triangleleft i$ and there exists $i'_\ell \in \text{Lev}_\ell^+([A_n, A^*]^t)$ such that $i' \triangleleft i'_0 \triangleleft i'_1 \triangleleft \dots \triangleleft i'_{N-1}$. This produces a chain C so that $|C^0| > N$ and $|C^1| > N$, contrary to homogeneity of chains.

Therefore, all antichains of P^{**} have size $\leq M$. By Theorem 2.3, P^{**} is the union of $\leq M$ chains. Therefore, $[A_n, A^*]^t$ is the union of $\leq N$ chains and $\leq M$ antichains. Putting this together, we see that P^t is the union of $\leq M(2N + 2)$ chains and $\leq N(2N + 2)$ antichains. \square

Lemma 3.10. *Under the assumption of Case 1, if $t < 2$ and $C \subseteq P^t$ is a chain, then there exists a uniform formula $\gamma_C(\bar{y})$, over at most $N \cdot (2(N + 1)^2 + N)$ elements of $\{\bar{b}_i : i \in P\}$, such that*

- (i) *for all $i \in C$, $\models \gamma_C(\bar{b}_i)$, and*
- (ii) *for all $i \in P$, if $\models \gamma_C(\bar{b}_i)$, then $i \in P^t$.*

That is, under Case 1, chains are roughly definable.

Proof. This is similar to the proof of Lemma 3.8, noting that we are assuming chains are homogeneous. Instead of having some small $C_0 \subseteq C$ which suffices to determine homogeneity for any $I_0 \subseteq P^{1-t}$ as in Corollary 3.7 for antichains, we will have to build C_n as we go along. By Lemma 2.7, we may assume that $C \subseteq [i, i']_P \subseteq P^t$ for some $i, i' \in C$ (this only decomposes C into at most N parts). We may also assume that $|C| > N$, or else we just use $\gamma_C(\bar{y}) = \bigvee_{i \in C} \bar{y} = \bar{b}_i$. For any $j \in P^{1-t}$, we have exactly one of three possibilities:

- (i) $\{i, j\}$ is an antichain for all $i \in C$,
- (ii) there exists $i_j^- \in C$ such that $i_j^- \triangleleft j$ and i_j^- is \triangleleft -maximal such, or

(iii) there exists $i_j^+ \in C$ such that $j \triangleleft i_j^+$ and i_j^+ is \triangleleft -minimal such.

In case (i), define $C'_j = \emptyset$. In case (ii), define C'_j to be the set of $N + 1$ C -consecutive elements $\trianglelefteq i_j^-$ and the $N + 1$ C -consecutive elements $\triangleright i_j^-$. In case (iii), define C'_j to be the set of $N + 1$ C -consecutive elements $\triangleleft i_j^+$ and the $N + 1$ C -consecutive elements $\triangleright i_j^+$. Define C_{-1} to be the set of $N + 1$ C -initial elements and $N + 1$ C -final elements. Now we begin our construction of $j_0, j_1, \dots \in P^{1-t}$ and $C_{-1} \subseteq C_0 \subseteq C_1 \subseteq \dots \subseteq C$ (with $|C_n| \leq 2(N + 1)(n + 1)$) as follows:

Suppose that there exists $j \in P^{1-t} - \{j_0, \dots, j_{n-1}\}$ such that $C_{n-1} \cup \{j_0, \dots, j_{n-1}, j\}$ is homogeneous. Let $j_n = j$ be any such and let $C_n = C_{n-1} \cup C'_j$. We claim that, in fact, $C_n \cup \{j_0, \dots, j_n\}$ is homogeneous. Given $i_0, \dots, i_N \in C_n \cup \{j_0, \dots, j_n\}$, to get $i'_0, \dots, i'_N \in C_{n-1} \cup \{j_0, \dots, j_n\}$ that are isomorphic (under $i_\ell \mapsto i'_\ell$), we fix the elements of $\{j_0, \dots, j_n\}$ and we push the elements inside C'_{j_n} away to the nearest elements of C_{n-1} . Then, the isomorphism yields

$$\text{tp}_\Delta(\bar{b}_{i_0}, \dots, \bar{b}_{i_n}) = \text{tp}_\Delta(\bar{b}_{i'_0}, \dots, \bar{b}_{i'_n}),$$

as desired.

Therefore, this construction halts after at most N steps, producing $I_0 = \{j_0, \dots, j_{K-1}\}$ and $C^* = C_K$ for $K \leq N$. Notice that, for any $j \in P^{1-t} - I_0$, $C \cup I \cup \{j\}$ is homogeneous if and only if $C^* \cup I_0 \cup \{j\}$ is homogeneous, which always fails by maximality of I_0 . The remainder of this proof follows exactly as the proof of Lemma 3.8. \square

We now prove Theorem 3.2 (i) \Rightarrow (ii) under Case 1.

By Lemma 3.9, there exists $t < 2$, chains C_n for $n < M(2N + 2)$ and antichains A_m for $m < N(2N + 2)$ so that $P^t = \bigcup_n C_n \cup \bigcup_m A_m$. For each n , let $\gamma_n^* = \gamma_{C_n}$ given by Lemma 3.10. For each m , let $\gamma_m^{**} = \gamma_{A_m}$ given by Lemma 3.8. Then take

$$\gamma(\bar{y}) = \bigvee_n \gamma_n^*(\bar{y}) \vee \bigvee_m \gamma_m^{**}(\bar{y}).$$

Then, for any $i \in P$, $\models \gamma(\bar{b}_i)$ if and only if $i \in P^t$ if and only if $\models \varphi(\bar{a}; \bar{b}_i)^t$. Thus, the formula γ^t defines the φ -type $p = \text{tp}_\varphi(\bar{a}/\{\bar{b}_i : i \in P\})$ in a uniform manner, as desired.

3.4. When Chains are Not Homogeneous

For this subsection, we assume:

Case 2. There exists $i_0 \triangleleft \dots \triangleleft i_N$ from P and $\sigma \in S_{N+1}$ such that

$$\text{tp}_\Delta(\bar{b}_{i_0}, \dots, \bar{b}_{i_N}) \neq \text{tp}_\Delta(\bar{b}_{i_{\sigma(0)}}, \dots, \bar{b}_{i_{\sigma(N)}}).$$

That is, chains are not homogeneous.

Definition 3.11. Fix $X \subseteq P$ and \leq_X a linear order on X . We say that $(X; \leq_X)$ is *order homogeneous* if, for any $i_0 <_X \dots <_X i_N$ from X and $j_0 <_X \dots <_X j_N$ from X , we have that

$$\text{tp}_\Delta(\bar{b}_{i_0}, \dots, \bar{b}_{i_N}) = \text{tp}_\Delta(\bar{b}_{j_0}, \dots, \bar{b}_{j_N}).$$

In other words, $\langle \bar{b}_i : i \in X \rangle$ is Δ -indiscernible with respect to $(X; \leq_X)$.

For example, for any chain $C' \subseteq P$, $(C'; \triangleleft|_{C'})$ is order homogeneous. The following lemma follows from Lemma 3.3 on the partial order $(X; \leq_X)$:

Lemma 3.12. *For any $X \subseteq P$ and \leq_X such that $(X; \leq_X)$ is order homogeneous, there does not exist $i_0 <_X i_1 <_X \dots <_X i_{2N+1}$ from X and $s < 2$ such that $i_n \in X^s$ if and only if n is even.*

Under the assumption of Case 2, for any $(X; \leq_X)$ that is order homogeneous with $|X| \geq N+1$, X is not homogeneous. Therefore, by Lemma 1.3 of [1], there exists $\ell < N$ and $\delta \in \pm\Delta$ such that, for all $i_0 <_X \dots <_X i_N$ from X , we have that

$$\models \delta(\bar{b}_{i_0}, \dots, \bar{b}_{i_N}) \wedge \neg \delta(\bar{b}_{i_0}, \dots, \bar{b}_{i_{\ell-1}}, \bar{b}_{i_{\ell+1}}, \bar{b}_{i_\ell}, \bar{b}_{i_{\ell+2}}, \dots, \bar{b}_{i_N}). \quad (2)$$

That is, δ is order-sensitive at ℓ .

Lemma 3.13. *Suppose that $i_0 \triangleleft \dots \triangleleft i_N$ are from P and $j \in P$ is such that one of the following conditions hold:*

- (i) $\{i_s, j\}$ is an antichain for all $s \leq N$,
- (ii) for some $n \neq \ell$, $i_n \triangleleft j$ and $\{i_s, j\}$ is an antichain for all $s > n$, or
- (iii) for some $n \neq \ell+1$, $j \triangleleft i_n$ and $\{i_s, j\}$ is an antichain for all $s < n$.

Then the ordering $i_0 < i_1 < \dots < i_\ell < j < i_{\ell+1} < \dots < i_N$ is not order homogeneous.

Proof. Conditions (i), (ii), and (iii) insure that the map $i_s \mapsto i_s$ for all $s \neq \ell, \ell+1$, $j \mapsto j$, and $i_\ell \mapsto i_{\ell+1}$ is an isomorphism of S -substructures. Therefore, by Δ -indiscernibility,

$$\text{tp}_\Delta(\bar{b}_{i_0}, \dots, \bar{b}_{i_{\ell-1}}, \bar{b}_j, \bar{b}_{i_{\ell+1}}, \bar{b}_{i_{\ell+2}}, \dots, \bar{b}_{i_N}) = \text{tp}_\Delta(\bar{b}_{i_0}, \dots, \bar{b}_{i_{\ell-1}}, \bar{b}_j, \bar{b}_{i_\ell}, \bar{b}_{i_{\ell+2}}, \dots, \bar{b}_{i_N}). \quad (3)$$

However, for tautological reasons,

$$\models \delta(\bar{b}_{i_0}, \dots, \bar{b}_{i_{\ell-1}}, \bar{b}_j, \bar{b}_{i_{\ell+1}}, \bar{b}_{i_{\ell+2}}, \dots, \bar{b}_{i_N}) \vee \neg \delta(\bar{b}_{i_0}, \dots, \bar{b}_{i_{\ell-1}}, \bar{b}_j, \bar{b}_{i_\ell}, \bar{b}_{i_{\ell+2}}, \dots, \bar{b}_{i_N}).$$

By (3), we get

$$\models \delta(\bar{b}_{i_0}, \dots, \bar{b}_{i_{\ell-1}}, \bar{b}_j, \bar{b}_{i_\ell}, \bar{b}_{i_{\ell+2}}, \dots, \bar{b}_{i_N}) \vee \neg \delta(\bar{b}_{i_0}, \dots, \bar{b}_{i_{\ell-1}}, \bar{b}_j, \bar{b}_{i_{\ell+1}}, \bar{b}_{i_{\ell+2}}, \dots, \bar{b}_{i_N}).$$

If $i_0 < i_1 < \dots < i_\ell < j < i_{\ell+1} < \dots < i_N$ were order homogeneous, this would imply that

$$\models \delta(\bar{b}_{i_0}, \dots, \bar{b}_{i_{\ell-1}}, \bar{b}_{i_{\ell+1}}, \bar{b}_i, \bar{b}_{i_{\ell+2}}, \dots, \bar{b}_{i_N}) \vee \neg \delta(\bar{b}_{i_0}, \dots, \bar{b}_{i_{\ell-1}}, \bar{b}_i, \bar{b}_{i_{\ell+1}}, \bar{b}_{i_{\ell+2}}, \dots, \bar{b}_{i_N}).$$

contrary to (2). \square

As a corollary, for any such $i_0 \triangleleft \dots \triangleleft i_N$ and $j \in P$, if $i \in P$ is such that $i_\ell \triangleleft i \triangleleft i_{\ell+1}$, then the formula

$$\theta(\bar{y}) = \delta(\bar{b}_{i_0}, \dots, \bar{b}_{i_{\ell-1}}, \bar{y}, \bar{b}_{i_{\ell+1}}, \bar{b}_{i_{\ell+2}}, \dots, \bar{b}_{i_N}) \wedge \neg \delta(\bar{b}_{i_0}, \dots, \bar{b}_{i_{\ell-1}}, \bar{y}, \bar{b}_{i_\ell}, \bar{b}_{i_{\ell+2}}, \dots, \bar{b}_{i_N}) \quad (4)$$

holds for \bar{b}_i and fails for \bar{b}_j . We use this to prove the following result for chains:

Lemma 3.14. *Under the assumption of Case 2, if $t < 2$ and $C \subseteq P^t$ is a chain, then there exists a uniform formula $\gamma_C(\bar{y})$, over at most $N \cdot (2(N+1)^2 + N)$ elements of $\{\bar{b}_i : i \in P\}$, such that*

(i) *for all $i \in C$, $\models \gamma_C(\bar{b}_i)$, and*

(ii) *for all $i \in P$, if $\models \gamma_C(\bar{b}_i)$, then $i \in P^t$.*

That is, under Case 2, chains are roughly definable.

Proof. As in the proof of Lemma 3.10, we can assume $C \subseteq [i, i']_P \subseteq P^t$ for some $i, i' \in C$ and $|C| > N$. Let $X_{-1} = C$ and $\leq_{-1} = \leq|_C$. Suppose that $n \geq 0$ and $(X_{n-1}; \leq_{n-1})$ is constructed. Suppose there exists $j \in P^{1-t}$ and \leq a linear order on $X_{n-1} \cup \{j\}$ extending \leq_{n-1} such that

- (a) $(X_{n-1} \cup \{j\}; \leq)$ is order homogeneous,
- (b) for all $j^* \in (X_{n-1} - C) \cup \{-\infty, \infty\}$, there exists at least $N + 1$ elements from C \leq -between j^* and j ,
- (c) if $(\exists i \in C)(j \triangleleft i)$, then, for all $i \in C$, $j < i$ if and only if $j \triangleleft i$, and
- (d) if $(\exists i \in C)(i \triangleleft j)$, then, for all $i \in C$, $i < j$ if and only if $i \triangleleft j$.

Then set $X_n = X_{n-1} \cup \{j\}$, and $\leq_n = \leq$. If no such j and \leq exists, then set $K = n$, $X^* = X_{K-1}$, and $\leq^* = \leq_{K-1}$ and the construction halts. We claim that this construction halts after N steps (i.e., $K \leq N$).

If not, fix $j_0 <^* \dots <^* j_N$ from $X^* - C$ (this exists since $|X^* - C| = K > N$). By construction, for each $n \leq N$, there exists (at least $N + 1$ many) i_n such that $j_{n-1} \leq i_n \leq j_n$ (where $j_{-1} = -\infty$). So we have $i_0 <^* j_0 <^* i_1 <^* \dots <^* i_N <^* j_N$. However, $i_n \in C \subseteq P^t$ and $j_n \in (X^* - C) \subseteq P^{1-t}$, so this contradicts Lemma 3.12. So $K \leq N$. We now show how to define γ_C from Lemma 3.10 using at most $2(N + 1)^2 + N$ elements from C .

Let C_{-1}^+ be the \triangleleft -initial $N + 1$ elements of C and let C_K^- be the \triangleleft -final $N + 1$ elements of C . Enumerate $X^* - C = \{j_0, \dots, j_{K-1}\}$ such that $j_0 <^* \dots <^* j_{K-1}$ and let C_n^- be the \triangleleft -final $N + 1$ elements $i \in C$ so that $i <^* j_n$ and let C_n^+ be the \triangleleft -initial $N + 1$ elements $i \in C$ so that $j_n <^* i$. Finally, let $C_0 = C_{-1}^+ \cup \bigcup_{n < K} (C_n^- \cup C_n^+) \cup C_K^+$. By construction, $|C_0| \leq 2(N + 1)^2$. For each $n \leq K$, consider

$$G_n = \{i \in C : (\exists i_{n-1} \in C_{n-1}^+, i_n \in C_n^-)(i_{n-1} \triangleleft i \triangleleft i_n)\},$$

the gap of C between C_{n-1}^+ and C_n^- . It is clear that

$$C = \bigcup_{n \leq K} (C_{n-1}^+ \cup G_n \cup C_n^+),$$

so, to define γ_C over $C_0 \cup (X^* - C)$ (which has $\leq 2(N + 1)^2 + N$ elements), we need only distinguish elements from $P^{1-t} - X^*$ and G_n for each n .

Fix $n \leq K$ and $j \in P^{1-t} - X^*$. As before, we have three cases to consider:

- (i) $\{i, j\}$ is an antichain for all $i \in (C_{n-1}^+ \cup G_n \cup C_n^+)$,
- (ii) there exists $i \in (C_{n-1}^+ \cup G_n \cup C_n^+)$ such that $i \triangleleft j$, or
- (iii) there exists $i \in (C_{n-1}^+ \cup G_n \cup C_n^+)$ such that $j \triangleleft i$.

If (i) holds, then by Lemma 3.13, the formula $\theta(\bar{y})$ as in (4) separates $i \in G_n$ from j as in case (i).

If (ii) holds, then let $I^- = \{i \in C : i \triangleleft j\}$ and let $I^+ = \{i \in C : i \not\triangleleft j\}$. If $C_{n-1}^+ \not\subseteq I^-$, then fix $i_\ell \in C_{n-1}^+ - I^-$, $i_0 \triangleleft \dots \triangleleft i_{\ell-1}$ from C_{n-1}^+ arbitrary such that $i_{\ell-1} \triangleleft i_\ell$, and $i_{\ell+1} \triangleleft \dots \triangleleft i_N$ from C_n^- arbitrary. Again by Lemma 3.13, we see that the formula

$$\delta(\bar{b}_{i_0}, \dots, \bar{b}_{i_{\ell-1}}, \bar{b}_{i_{\ell+1}}, \bar{y}, \bar{b}_{i_{\ell+2}}, \dots, \bar{b}_{i_N}) \vee \neg \delta(\bar{b}_{i_0}, \dots, \bar{b}_{i_{\ell-1}}, \bar{b}_{i_\ell}, \bar{y}, \bar{b}_{i_{\ell+2}}, \dots, \bar{b}_{i_N})$$

holds of \bar{b}_i for any $i \in G_n$ and fails for \bar{b}_j . Therefore, we may assume $C_{n-1}^+ \subseteq I^-$. Similarly, we may assume $C_n^- \subseteq I^+$. Therefore, if we let \leq be the extension of \leq^* setting $i < j$ for all $i \in I^-$ and $j < i$ for all $i \in I^+$, conditions (b), (c), and (d) of the construction holds for j . If (a) holds, then we contradict the fact that the construction halted, so we may assume that (a) fails. Therefore, there exists $i_0 \triangleleft \dots \triangleleft i_s$ from $(C_{n-1}^+ \cup G_n \cup C_n^-) \cap I^-$ and $i_{s+1} \triangleleft \dots \triangleleft i_N$ from $(C_{n-1}^+ \cup G_n \cup C_n^-) \cap I^+$ so that

$$\text{tp}_\Delta(\bar{b}_{i_0}, \dots, \bar{b}_{i_{s-1}}, \bar{b}_{i_s}, \bar{b}_{i_{s+1}}, \dots, \bar{b}_{i_N}) \neq \text{tp}_\Delta(\bar{b}_{i_0}, \dots, \bar{b}_{i_{s-1}}, \bar{b}_j, \bar{b}_{i_{s+1}}, \dots, \bar{b}_{i_N}).$$

However, since $C_{n-1}^+ \subseteq I^-$ we may choose $i_0 \triangleleft \dots \triangleleft i_{s-1}$ in C_{n-1}^+ (by Δ -indiscernibility) and we may similarly choose $i_{s+1} \triangleleft \dots \triangleleft i_N$ in C_n^- . Also, this is witnessed by some $\delta \in \pm\Delta$. Therefore, we have that

$$\models \delta(\bar{b}_{i_0}, \dots, \bar{b}_{i_{s-1}}, \bar{b}_i, \bar{b}_{i_{s+1}}, \dots, \bar{b}_{i_N}) \wedge \neg \delta(\bar{b}_{i_0}, \dots, \bar{b}_{i_{s-1}}, \bar{b}_j, \bar{b}_{i_{s+1}}, \dots, \bar{b}_{i_N})$$

for all $i \in G_n$. Hence, this formula separates $i \in G_n$ and j as in case (ii). Case (iii) follows by symmetry. \square

Now that we can roughly define chains, we have to deal with the space between two antichains.

Lemma 3.15. *Suppose that $A \trianglelefteq A'$ two maximal antichains of P and $t < 2$ are such that $|A^t| > N$ and $|(A')^t| > N$. Then, there exists a uniform $\gamma_{A,A'}(\bar{y})$ over at most $2^{(2N+2)} \cdot (2N + 2(N+1)^2)$ elements of $\{\bar{b}_i : i \in P\}$ such that*

(i) *for all $i \in [A, A']^t$, $\models \gamma_{A,A'}(\bar{b}_i)$, and*

(ii) *for all $i \in P - [A, A']$, if $\models \gamma_{A,A'}(\bar{b}_i)$, then $i \in P^t$.*

That is, $\gamma_{A,A'}$ roughly defines $[A, A']^t$ over $P - [A, A']$.

Proof. Fix J_0 and J_1 and $X_{J,J'}$ for each $J \subseteq J_0$ and $J' \subseteq J_1$ as in Lemma 3.6. Note that $|J_0| \leq 2N+1$ and $|J_1| \leq 2N+1$. Fixing $J \subseteq J_0$ and $J' \subseteq J_1$, we now focus roughly defining $X_{J,J'}^t$ (and there are, at most, $2^{(2N+2)}$ of these sets).

Let $A_s = \text{Lev}_s^-(X_{J,J'}^t)$ for $s < \ell$, let $X^* = X_{J,J'}^t - \bigcup_{s \leq \ell} A_s$, let $A_{N-s} = \text{Lev}_s^+(X^*)$ for $s < N - \ell - 1$, and let $X^{**} = X^* - \bigcup_{\ell+1 \leq s \leq N} A_s$. Each of A_s for $s \leq N$ is roughly definable by γ_{A_s} by Lemma 3.8. Therefore, if $X^{**} = \emptyset$, we are done, so suppose not. Fix A^- a \triangleleft -antichain of X^{**} that is \triangleleft -minimal and $|A^-| = N+1$. Similarly, fix A^+ a \triangleleft -antichain of X^{**} that is \triangleleft -maximal and $|A^+| = N+1$. If these do not exist, then X^{**} is a union of $\leq N$ chains, say C_ℓ . Use $\bigvee_\ell \gamma_{C_\ell}$ as in Lemma 3.14 to roughly define X^{**} . So we may assume A^- and A^+ exist. By choice of minimality of A^- , for any $i \in X^{**}$, either $A^- \cup \{i\}$ is an antichain or $A^- \triangleleft i$. The same holds for A^+ (except with the reverse ordering).

Now fix $I^- \subseteq P^{1-t} - [A, A']$ maximal so that $A^- \cup I^-$ is homogeneous and fix $I^+ \subseteq P^{1-t} - [A, A']$ maximal so that $A^+ \cup I^+$ is homogeneous. By Lemma 3.5, $|I^-| \leq N$ and $|I^+| \leq N$. For all $i \in A^-$, choose $\iota_{i,0} \triangleleft \dots \triangleleft \iota_{i,\ell-1} \triangleleft \iota_{i,\ell} = i$ from $X_{J,J'}^t$ (which exists by construction of X^{**}) and let $X^- = \{\iota_{i,s} : i \in A^-, s \leq \ell\}$. Similarly construct X^+ with chains $i = \iota'_{i,\ell+1} \triangleleft \dots \triangleleft \iota'_{i,N}$ for $i \in A^+$. Let $X_0 = X^- \cup X^+ \cup I^- \cup I^+$ and notice that $|X_0| = 2N + (N+1)^2$.

Now, fix any $j \in P^{1-t} - [A, A']$ and any $i \in X^{**}$. We claim that j does not have the same Δ -type as i over X_0 . That is, there exists $i_1, \dots, i_N \in X_0$ distinct such that

$$\text{tp}_\Delta(\bar{b}_i; \bar{b}_{i_1}, \dots, \bar{b}_{i_N}) \neq \text{tp}_\Delta(\bar{b}_j; \bar{b}_{i_1}, \dots, \bar{b}_{i_N}).$$

First, if $A^- \cup \{i\}$ is an antichain, then by Lemma 3.6, $A^- \cup I^-$ is homogeneous if and only if $A^- \cup \{i\} \cup I^-$ is homogeneous. Therefore, if j had the same Δ -type as i over X_0 ($X_0 \supseteq A^- \cup I^-$), then we would have that $A^- \cup \{j\} \cup I^-$ is homogeneous, contrary to the maximality of I^- . Thus they do not have the same Δ -type. We show this when $A^+ \cup \{i\}$ is an antichain by symmetry. So we may assume $A^- \triangleleft i \triangleleft A^+$. Therefore, we have $i_0 \triangleleft \dots \triangleleft i_\ell \triangleleft i \triangleleft i_{\ell+1} \triangleleft \dots \triangleleft i_N$ with $i_0, \dots, i_N \in X_0$ (by construction of X_0). Hence, by Lemma 3.13, j cannot have the same Δ -type as i over X_0 . Therefore, we can separate $i \in X^{**}$ from $j \in P^{1-t} - [A, A']$ with a formula over X_0 . Let

$$\gamma_{X^{**}}(\bar{y}) = \bigvee_{i \in X^{**}} \bigwedge \{ \delta(\bar{y}, \bar{b}_{i_1}, \dots, \bar{b}_{i_N}) : i_1, \dots, i_N \in X_0, \models \delta(\bar{b}_i, \bar{b}_{i_1}, \dots, \bar{b}_{i_N}) \}.$$

Note that, *a priori*, $\gamma_{X^{**}}$ ranges over arbitrarily many elements $i \in X^{**}$. However, there are only boundedly many Δ -types over X_0 , so this is a uniform formula over $2N + (N + 1)^2$ elements of $\{\bar{b}_i : i \in P\}$. By construction, for all $i \in X^{**}$, $\models \gamma_{X^{**}}(\bar{b}_i)$. Furthermore, for all $j \in P^{1-t} - [A, A']$, $\models \neg \gamma_{X^{**}}(\bar{b}_j)$. This gives the desired result. \square

We now prove Theorem 3.2 (i) \Rightarrow (ii) under Case 2, completing the proof.

By Lemma 2.5, there exists $A_0 \triangleleft \dots \triangleleft A_{K-1}$ for $K \leq 2N + 2$ maximal antichains of P such that, for all $n \leq K$ and all antichains $A \subseteq [A_{n-1}, A_n]$, $|A^{n(\bmod 2)}| \leq M$ (let $A_{-1} = -\infty$ and $A_K = \infty$). For each $n \equiv 0(\bmod 2)$, let $A'_n \subseteq [A_{n-1}, A_n]$ be a maximal antichain of P with $|(A'_n)^1| > M$ and choose $A'_n \triangleleft$ -maximal such. For each $n \equiv 1(\bmod 2)$, for any antichain $A \subseteq (A'_{n-1}, A_n)^1$, $|A| \leq M$ by construction. Therefore, by Theorem 2.3, $(A'_{n-1}, A_n)^1 = \bigcup_{\ell < M} C_{n,\ell}$ for chains $C_{n,\ell} \subseteq P^1$. Likewise, for $n \equiv 0(\bmod 2)$, $[A_{n-1}, A'_n]^0 = \bigcup_{\ell < M} C_{n,\ell}$ for chains $C_{n,\ell} \subseteq P^0$. Let:

$$\psi(\bar{y}) = \bigvee_{n \equiv 0(\bmod 2)} \left(\gamma_{A_{n-1}, A'_n}(\bar{y}) \wedge \bigwedge_{\ell < M} \neg \gamma_{C_{n,\ell}}(\bar{y}) \right) \vee \bigvee_{n \equiv 1(\bmod 2), \ell < M} \gamma_{C_{n,\ell}}(\bar{y})$$

for $\gamma_{A,A'}$ as in Lemma 3.15 and γ_C as in Lemma 3.14. Then, $\models \varphi(\bar{a}; \bar{b}_i)$ if and only if $i \in P^1$ if and only if $i \in C_{n,\ell}$ for some $n \equiv 1(\bmod 2)$ and some $\ell < M$ or $i \in [A_{n-1}, A'_n] - \bigcup_{\ell < M} C_{n,\ell}$ for some $n \equiv 0(\bmod 2)$. This holds if and only if $\models \psi(\bar{b}_i)$. Since all of the formulas $\gamma_{A,A'}$ and γ_C are uniform, ψ is uniform. This concludes the proof of Theorem 3.2. As mentioned before, Theorem 1.1 follows as a corollary.

4. Discussion

With Theorem 1.1 in hand, one is tempted to solve the general UDTFS Conjecture by the following means: Prove all finite sets can be made into a partial order indiscernible. Unfortunately, there are simple examples to show that this is not true even when we assume that φ has independence dimension 1.

Example 4.1. Consider $X = 5 = \{0, 1, 2, 3, 4\}$ and let

$$Y = \{\{0\}, \{0, 1, 2\}, \{2, 3, 4\}, \{4\}\},$$

a subset of the powerset of X . Let $R(x, y)$ be a binary relation that holds if and only if $x \in X$, $y \in Y$, and $x \in y$. The relation $R(x, y)$ clearly

has independence dimension 1 but we claim that there is no partial order \trianglelefteq on Y so that $\langle y : y \in Y \rangle$ is a $\Delta_{1,R}$ -indiscernible sequence. To see this, suppose there was such a \trianglelefteq . First notice that $\{0\}$ and $\{0, 1, 2\}$ cannot be an antichain since they are not homogeneous (i.e., $\text{tp}_\Delta(\{0\}, \{0, 1, 2\}) \neq \text{tp}_\Delta(\{0, 1, 2\}, \{0\})$ for $\Delta = \Delta_{1,R}$ since $\{0\} \subseteq \{0, 1, 2\}$ and not vice-versa). Therefore, $\{0\} \triangleleft \{0, 1, 2\}$ or $\{0, 1, 2\} \triangleleft \{0\}$. Now $\{0, 1, 2\}$ and $\{2, 3, 4\}$ must form an antichain as the Δ -type of the pair is unequal to the Δ -type of $(\{0\}, \{0, 1, 2\})$ or $(\{0, 1, 2\}, \{0\})$. Similarly, $\{0\}$ and $\{4\}$ must form an antichain, and therefore $\text{tp}_\Delta(\{0, 1, 2\}, \{2, 3, 4\}) = \text{tp}_\Delta(\{0\}, \{4\})$. However, this cannot hold; for example, the first pair intersect non-trivially while the second pair does not. Therefore, this is a contradiction.

The problem, of course, is distinguishing between the two types of incomparability when using the set-inclusion ordering. Assuming independence dimension ≤ 1 , if two sets are incomparable, then either they are disjoint or their union is the whole space. This can be remedied by considering instead an index language $S = \{\trianglelefteq, E\}$ where E is a binary relation symbol. Then, one can use E on incomparable elements to distinguish the two types of incomparability. This leads to the following open question:

Open Question 4.2. *Do all dependent formulas have uniform definability of types over indiscernible sequences indexed by finite S -structures P so that \trianglelefteq^P is a partial order and E^P is a symmetric binary relation on incomparable elements?*

An alternative solution is to only deal with formulas φ of independence dimension ≤ 1 that are directed in the sense of [5].

Of course, as mentioned in the introduction, we would like to expand this notion of definability of types to even more general index structures. For example:

Open Question 4.3. *Do all dependent formulas have uniform definability of types over indiscernible sequences indexed by finite directed graphs?*

One problem with directed graphs is that, without transitivity, there is no notion of minimal elements. All means of obtaining UDTFS both in this paper and in [1] use the fact that finite partial orders have minimal elements. Thus it seems that an entirely new approach would be needed to answer the question for directed graphs.

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